AEI 2002-029

Monodromy of solutions of the Knizhnik-Zamolodchikov equation: $SL(2,\mathbb{C})/SU(2)$ WZNW Model

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Abstract

Three explicit and equivalent representations for the monodromy of the conformal blocks in the $SL(2,\mathbb{C})/SU(2)$ WZNW model are proposed in terms of the same quantity computed in Liouville field theory. We show that there are two possible fusion matrices in this model. This is due to the fact that the conformal blocks, being solutions to the Knizhnik-Zamolodchikov equation, have a singularity when the $SL(2,\mathbb{C})$ isospin coordinate x equals the worldsheet variable z. We study the asymptotic behaviour of the conformal block when x goes to z. The obtained relation inserted into a four point correlation function in the $SL(2,\mathbb{C})/SU(2)$ WZNW gives some expression in terms of two correlation functions in Liouville field theory.

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Introduction

The $SL(2,\mathbb{C})/SU(2)$ (or H_3^+) WZNW model is the second simplest non compact Conformal Field Theory (*i.e.* with a continuous spectrum of primary fields) besides Liouville field theory. This model plays a role in condensed matter physics, as it is believed to describe the plateau transitions in the Integer Quantum Hall effect [1]. It is also intensively studied in the context of string theory on AdS_3 (see for example [2] for a non exhaustive list of references). As its characteristics are closely related to those of Liouville field theory, one may wonder to what extent it is possible to apply to the H_3^+ WZNW model the techniques developped for Liouville field theory (computation of the bulk three point function: [3,4], bulk one point function in presence of a boundary and boundary two point function [5], Liouville field theory on the pseudosphere [6], bulk-boundary function [7], boundary three point function [8], fusion coefficients and proof for crossing symmetry [9,10]). It turns out in the former model that the construction for the structure constants [11,2] can be achieved *mutatis mutandis* along the same lines. Less staightforward is a proof for crossing symmetry, as well as the construction of the fusion matrix. In Liouville field theory, the fusion matrix was constructed as Racah-Wigner coefficients for some well chosen continuous representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ [9,10]; in this case the proof for crossing symmetry boiled down to an orthogonality relation for these Racah-Wigner coefficients. In the case of the H_3^+ WZNW model, it was proposed in [12] as a quantum group structure the "pair" $U_q(sl(2)), U_{q'}(osp(1|2))$. We will not follow this approach here, as it turns out that there is a way to avoid quantum group methods to construct the fusion matrix: as it was noticed in [1] and [13], it is possible to adapt to the H_3^+ WZNW model some observation previously made by Zamolodchikov and Fateev in [14] in the context of the compact $SU(2)$ WZNW model: these authors observed that there exists a 5 point conformal block in the (k+2,1) Minimal Model that satisfies the same Knizhnik-Zamolodchikov equation as the 4 point conformal block in the $SU(2)$ WZNW model. The adaptation of this relation to the non compact case is straighforward and allowed to prove crossing symmetry in the H_3^+ WZNW model [13], as a consequence from a similar property of a 5 point function in Liouville field theory. Actually, this method also permits[2] to construct the monodromy of the conformal blocks in the H_3^+ WZNW model (*i.e.* the fusion matrix, also called fusion coefficients) from the knowledge of the monodromy for a special 5 point function in Liouville field theory. Let us emphasize on the fact that the fusion matrix is the most important quantity of a CFT on genus zero, as it encodes the information on degenerate representations, fusion rules, intermediate states appearing in a four point function; it also permits the computation of boundary structure constants and enters the consistency relations satisfied by structure constants [15,16].

This paper is organized as follows: in the first section we will recall some useful information about Liouville field theory and the H_3^+ WZNW model. In section two we will propose three explicit formulas for the monodromy of the conformal blocks in H_3^+ WZNW, constructed in terms of two Liouville fusion matrices. Let us mention that the case we have here is somewhat different to all other known cases so far (*i.e.* minimal models,

[2]This was first observed by J.Teschner.

$SU(2)$ WZNW model, Liouville field theory), as there are two possible fusion matrices for this model: this comes from the fact that the conformal blocks, in addition to the usual singularities at $z = 0, 1, \infty$, have also a singularity at $z = x$ (this plays an important role in [17]), where z is the worldsheet variable, and x the isospin variable (or space-time coordinate). When deriving the formula for the fusion matrix, there are two cases to consider, depending whether $Im(z - x)$ is positive or negative. This accounts for a phase factor in the formulas, which is different according to the sign of $Im(z - x)$. In section three, we will study the asymtotic behavior of the conformal blocks in the limit where x goes to z, and we will see that they can be expressed as a sum of two Liouville conformal blocks. The first term is regular in this limit, whereas the second term contains some singularity. This relation permits us to rewrite a four point correlation function in the limit x goes to z as a sum over two Liouville correlation functions. We finish by some concluding remarks; appendix A contains some particular cases of the Liouville fusion coefficients needed in the main text to derive our main result, in appendix B we recover some well known special cases from our H_3^+ WZNW fusion matrix. Finally, in appendix C we present a way to find degenerate representations and fusion rules for $\hat{sl}(2)$ from degenerate representations and fusion rules for the Virasoro algebra.

1 Requisites

1.1 Liouville field theory

Let the $V_\alpha(z, \bar{z})$ be the primary fields with conformal weight $\Delta_\alpha^L = \alpha(Q - \alpha)$ where $Q = b + b^{-1}$; b is the coupling constant in Liouville field theory that we shall call for short LFT. The central charge of the Virasoro algebra is $c = 1 + 6Q^2$.
Let

$$\langle V_{\alpha_4}(\infty) V_{\alpha_3}(1) V_{\alpha_2}(z, \bar{z}) V_{\alpha_1}(0) \rangle \equiv \mathcal{V}_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(z, \bar{z})$$

denote a four point correlation function in LFT and

$$\mathcal{F}^s_{\alpha_{21}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|z) \tag{1}$$

be the corresponding conformal block in the s-channel. The conformal block is completly determined by the conformal symmetry (although there is no known closed form for it in general), and is normalized such that

$$\mathcal{F}^s_{\alpha_{21}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|z) \sim_{z \to 0} z^{\Delta^L_{\alpha_{21}} - \Delta^L_{\alpha_1} - \Delta^L_{\alpha_2}} (1 + \mathcal{O}(z)) \tag{2}$$

Let us note that the Liouville conformal block depends on conformal weights only.
There exist [18] invertible fusion transformations between s- and t-channel conformal blocks, defining the Liouville fusion matrix:

$$\mathcal{F}^s_{\alpha_{21}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|z) = \int_{\frac{Q}{2} + i\mathbb{R}^+} d\alpha_{32} F^L_{\alpha_{21}\alpha_{32}} \left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right] \mathcal{F}^t_{\alpha_{32}}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|1 - z) \tag{3}$$

The explicit expression for the LFT fusion matrix was given in [9], in terms of a b-deformed ${}_4F_3$ hypergeometric function in the Barnes representation:

$$
\begin{aligned}
&F^L_{\alpha_{21}\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right] = \\
&\quad \frac{\Gamma_b(2Q-\alpha_3-\alpha_2-\alpha_{32})\Gamma_b(\alpha_3+\alpha_{32}-\alpha_2)\Gamma_b(Q-\alpha_2-\alpha_{32}+\alpha_3)\Gamma_b(Q-\alpha_3-\alpha_2+\alpha_{32})}{\Gamma_b(2Q-\alpha_1-\alpha_2-\alpha_{21})\Gamma_b(\alpha_1+\alpha_{21}-\alpha_2)\Gamma_b(Q-\alpha_2-\alpha_{21}+\alpha_1)\Gamma_b(Q-\alpha_2-\alpha_1+\alpha_{21})} \\
&\quad \frac{\Gamma_b(Q-\alpha_{32}-\alpha_1+\alpha_4)\Gamma_b(\alpha_{32}+\alpha_1+\alpha_4-Q)\Gamma_b(\alpha_1+\alpha_4-\alpha_{32})\Gamma_b(\alpha_4+\alpha_{32}-\alpha_1)}{\Gamma_b(Q-\alpha_{21}-\alpha_3+\alpha_4)\Gamma_b(\alpha_{21}+\alpha_3+\alpha_4-Q)\Gamma_b(\alpha_3+\alpha_4-\alpha_{21})\Gamma_b(\alpha_{21}+\alpha_4-\alpha_3)} \\
&\quad \frac{\Gamma_b(2Q-2\alpha_{21})\Gamma_b(2\alpha_{21})}{\Gamma_b(Q-2\alpha_{32})\Gamma_b(2\alpha_{32}-Q)}\frac{1}{i}\int_{-i\infty}^{i\infty} ds\ \frac{S_b(U_1+s)S_b(U_2+s)S_b(U_3+s)S_b(U_4+s)}{S_b(V_1+s)S_b(V_2+s)S_b(V_3+s)S_b(Q+s)}
\end{aligned}
\tag{4}
$$

with

$$
\begin{array}{ll}
U_1 = \alpha_{21}+\alpha_1-\alpha_2 & V_1 = Q+\alpha_{21}-\alpha_{32}-\alpha_2+\alpha_4\\
U_2 = Q+\alpha_{21}-\alpha_2-\alpha_1 & V_2 = \alpha_{21}+\alpha_{32}+\alpha_4-\alpha_2\\
U_3 = \alpha_{21}+\alpha_3+\alpha_4-Q & V_3 = 2\alpha_{21}\\
U_4 = \alpha_{21}-\alpha_3+\alpha_4 &
\end{array}
$$

The special function entering the formula is $\Gamma_b(x) \equiv \Gamma_{b^{-1}}(x) \equiv \frac{\Gamma_2(x|b,b^{-1})}{\Gamma_2(Q/2|b,b^{-1})}$, where $\Gamma_2(x)$ is the Double Gamma function introduced by Barnes [19], Γ_b satisfies the following functional relation

$$
\Gamma_b(x+b) = \frac{\sqrt{2\pi}b^{bx-\frac{1}{2}}}{\Gamma(bx)}\Gamma_b(x) \tag{5}
$$

Γ_b is a meromorphic function of x, which poles are located at $x = -nb - mb^{-1}, n, m \in \mathbb{N}$. The function $S_b(x)$ is defined as $S_b(x) \equiv \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}$.
Let us quote the following properties:

- The Liouville fusion matrix is holomorphic in the following range [9]

$$
\begin{array}{ll}
|\mathrm{Re}(\alpha_2+\alpha_3+\alpha_{32}-Q)| < Q & |\mathrm{Re}(\alpha_4+\alpha_1+\alpha_{32}-Q)| < Q\\
|\mathrm{Re}(\alpha_2+\alpha_3-\alpha_{32})| < Q & |\mathrm{Re}(\alpha_4+\alpha_1-\alpha_{32})| < Q\\
|\mathrm{Re}(\alpha_3+\alpha_{32}-\alpha_2)| < Q & |\mathrm{Re}(\alpha_4+\alpha_{32}-\alpha_1)| < Q\\
|\mathrm{Re}(\alpha_2+\alpha_{32}-\alpha_3)| < Q & |\mathrm{Re}(\alpha_1+\alpha_{32}-\alpha_4)| < Q
\end{array}
$$

- It satisfies the symmetry properties

$$
F^L_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right] = F^L_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc}\alpha_4 & \alpha_1\\ \alpha_3 & \alpha_2\end{array}\right] = F^L_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc}\alpha_2 & \alpha_3\\ \alpha_1 & \alpha_4\end{array}\right] \tag{6}
$$

- As the conformal blocks depends on conformal weights only, so does the Liouville fusion matrix, *i.e.* is invariant when one of the α_i is substituted by $Q - \alpha_i$.

The LFT fusion matrix was built in terms of the Racah-Wigner coefficients for an appropriate continuous series of representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$ with deformation parameter $q = e^{i\pi b^2}$ [9,10]. This construction ensures that the fusion matrix satisfies the set of Moore-Seiberg equations (or polynomial equations) [20].
Let us quote the *pentagonal* equation:

$$\int_{\frac{Q}{2}+i\mathbb{R}^+} d\delta_1 F^L_{\beta_1\delta_1}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \beta_2 & \alpha_1 \end{bmatrix} F^L_{\beta_2\gamma_2}\begin{bmatrix} \alpha_4 & \delta_1 \\ \alpha_5 & \alpha_1 \end{bmatrix} F^L_{\delta_1\gamma_1}\begin{bmatrix} \alpha_4 & \alpha_3 \\ \gamma_2 & \alpha_2 \end{bmatrix} = F^L_{\beta_2\gamma_1}\begin{bmatrix} \alpha_4 & \alpha_3 \\ \alpha_5 & \beta_1 \end{bmatrix} F^L_{\beta_1\gamma_2}\begin{bmatrix} \gamma_1 & \alpha_2 \\ \alpha_5 & \alpha_1 \end{bmatrix} \tag{7}$$

and the *two hexagonal* equations:

$$F^L_{\alpha_{21},\beta}\begin{bmatrix} \alpha_4 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{bmatrix} e^{i\pi\epsilon(\Delta^L_{\alpha_1}+\Delta^L_{\alpha_2}+\Delta^L_{\alpha_3}+\Delta^L_{\alpha_4}-\Delta^L_{\alpha_{21}}-\Delta^L_{\beta})} = \int_{\frac{Q}{2}+i\mathbb{R}^+} d\alpha_{32} F^L_{\alpha_{21},\alpha_{32}}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} F^L_{\alpha_{32},\beta}\begin{bmatrix} \alpha_2 & \alpha_4 \\ \alpha_3 & \alpha_1 \end{bmatrix} e^{i\pi\epsilon\Delta_{\alpha_{32}}} \tag{8}$$

where $\epsilon = \pm$.
We will also need the Liouville three point function; an explicit formula for it was proposed in [3,4]

$$C^L(\alpha_3,\alpha_2,\alpha_1) = \left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{\frac{Q-\alpha_1-\alpha_2-\alpha_3}{b}} \frac{\Upsilon_0\Upsilon_b(2\alpha_1)\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)\Upsilon_b(\alpha_1+\alpha_3-\alpha_2)\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)} \tag{9}$$

where $\Upsilon_b(x)^{-1} = \Gamma_b(x)\Gamma_b(Q-x), \gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$, $\Upsilon_0 = \mathrm{res}_{x=0}\frac{d\Upsilon_b(x)}{dx}$, and μ is the cosmological constant.
A four point function in LFT is written (if the $Re(\alpha_i), i = 1\ldots4$ are close enough to $Q/2$)

$$\mathcal{V}_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(z,\bar{z}) = = \frac{1}{2}\int_{\frac{Q}{2}+i\mathbb{R}} d\alpha_{21} C^L(\alpha_4,\alpha_3,\alpha_{21}) C^L(Q-\alpha_{21},\alpha_2,\alpha_1)|\mathcal{F}^s_{\alpha_{21}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4|z)|^2 \tag{10}$$

1.2 $SL(2,\mathbb{C})/SU(2)$ WZNW model

Let us denote k the level of the current algebra; k is formally related to the coupling constant b in LFT by $k \equiv b^{-2} + 2$. The central charge of the theory is $c = \frac{3k}{k-2}$. The

primary fields $\Phi^j(x|z)$ have conformal weight $\Delta_j = -b^2 j(j+1)$.
The action of the $SL(2,\mathbb{C})$ currents on the primary fields is given by

$$J^a(z)\Phi^j(x|w) \sim \frac{D^a_j}{z-w}\Phi^j(x|w), \quad a = \pm, 3 \qquad \bar{J}^a(z)\Phi^j(x|w) \sim \frac{\bar{D}^a_j}{\bar{z}-\bar{w}}\Phi^j(x|w) \tag{11}$$

where D^a_j are differential operators representing the $sl(2)$ algebra

$$D^+_j = \frac{\partial}{\partial x}, \qquad D^3_j = x\frac{\partial}{\partial x} + j, \qquad D^-_j = x^2\frac{\partial}{\partial x} + 2jx, \tag{12}$$

the $\bar{D}^a_j$ their complex conjugates.
Let

$$\left\langle \Phi^{j_4}(\infty|\infty)\Phi^{j_3}(1|1)\Phi^{j_2}(x|z)\Phi^{j_1}(0|0)\right\rangle \equiv \Phi_{j_4,j_3,j_2,j_1}(x,\bar{x}|z,\bar{z})$$

be a four point correlation function in the $SL(2,\mathbb{C})/SU(2)$ WZNW model and $\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z)$ be the corresponding s-channel conformal block. It is uniquely defined as the solution of the Knizhnik-Zamolodchikov equation $(z(z-1)\partial_z + b^2\mathcal{D}^{(2)}_x)\mathcal{G}^s(x|z) = 0$, where

$$\begin{aligned}\mathcal{D}^{(2)}_x &= x(x-1)(x-z)\partial_x^2 \\ &\quad -[(\kappa-1)(x^2-2zx+z) + 2j_1x(z-1) + 2j_2x(x-1) + 2j_3z(x-1)]\partial_x \\ &\quad +2j_2\kappa(x-z) + 2j_1j_2(z-1) + 2j_2j_3z\end{aligned} \tag{13}$$

where $\kappa = j_1 + j_2 + j_3 - j_4$, and the normalization prescription

$$\mathcal{G}^s(x|z) \sim z^{\Delta_{j_{21}}-\Delta_{j_2}-\Delta_{j_1}}x^{j_1+j_2-j_{21}}(1+\mathcal{O}(x)+\mathcal{O}(z)). \tag{14}$$

in the limit of taking first $z \to 0$, then $x \to 0$. Let us note that the KZ equation has four singular points, located at $z = 0, 1, x, \infty$, and that there is no closed form known for the conformal blocks in general.
The monodromy of the conformal blocks (or fusion matrix of the H_3^+ WZNW model) is defined as

$$\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z) = \int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj_{32}F^{H_3^+}_{j_{21}j_{32}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1\end{bmatrix}\mathcal{G}^t_{j_{32}}(j_1,j_2,j_3,j_4|1-x,1-z) \tag{15}$$

It is our aim to compute this quantity.
We will also need the expression for the three point function [11]

$$\begin{aligned}C^{H_3^+}(j_3,j_2,j_1) &= (\nu(b)b^{-2b^2})^{1+j_1+j_2+j_3} \\ &\frac{C_0(b)\Upsilon_b(-2bj_1)\Upsilon_b(-2bj_2)\Upsilon_b(-2bj_3)}{\Upsilon_b(-bj_1-bj_2-bj_3-b)\Upsilon_b(-bj_1-bj_2+bj_3)\Upsilon_b(-bj_1-bj_3+bj_2)\Upsilon_b(-bj_2-bj_3+bj_1)}\end{aligned} \tag{16}$$

It is known [11] that the expression for a four point correlation function is (if the $Re(j_i), i = 1\dots 4$ are close enough to $-\frac{1}{2}$).

$$\Phi_{j_4,j_3,j_2,j_1}(x,\bar{x}|z,\bar{z}) = \int_{-\frac{1}{2}+i\mathbb{R}} dj_{21} B(j_{21}) C^{H_3^+}(j_4,j_3,j_{21}) C^{H_3^+}(j_{21},j_2,j_1) |\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z)|^2 \tag{17}$$

where

$$B(j) = (\nu(b))^{-2j-1}\gamma(1+b^2(2j+1)).$$

2 Monodromy of solutions of the KZ equation

2.1 Method

It follows from a remarkable observation of [14] straightforwardly adapted to the non-compact case that the conformal block of a special 5 point function in LFT satisfies *the same KZ equation* as the conformal block in the H_3^+ WZNW model (to my knowledge there is no deep explanation for this phenomenon); hence, knowing the precise correspondence between the two quantities allows us to compute the monodromy of the conformal block in the H_3^+ WZNW model in terms of the monodromy of this 5 point LFT conformal block. Let $\mathcal{F}^s_{\alpha_{21}}(\alpha_1,\alpha_2,-\frac{1}{2b},\alpha_3,\alpha_4|x,z)$ be the 5 point conformal block corresponding to the LFT correlation function:

$$\left\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{-\frac{1}{2b}}(x,\bar{x})V_{\alpha_2}(z,\bar{z})V_{\alpha_1}(0)\right\rangle$$

it was shown in [14] that

$$\mathcal{F}^s_{\alpha_{21}}(\alpha_1,\alpha_2,-\frac{1}{2b},\alpha_3,\alpha_4|x,z) = x^{b^{-1}\alpha_1}(1-x)^{b^{-1}\alpha_3}(x-z)^{b^{-1}\alpha_2} z^{\frac{1}{2}\gamma_{12}}(1-z)^{\frac{1}{2}\gamma_{13}}\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z) \tag{18}$$

where the parameters of the two theories are related by

$$\begin{array}{ll}
2\alpha_1 = -b(j_1+j_2-j_3-j_4-b^{-2}-1) & 2\alpha_3 = -b(-j_1+j_2+j_3-j_4-b^{-2}-1) \\
2\alpha_2 = -b(j_1+j_2+j_3+j_4+1) & 2\alpha_4 = -b(-j_1+j_2-j_3+j_4-b^{-2}-1) \\
\alpha_{21} = -bj_{21}+\frac{1}{2b} & \alpha_{32} = -bj_{32}+\frac{1}{2b} \\
\gamma_{12} = 4b^2 j_1 j_2 - 4\alpha_1\alpha_2 & \gamma_{23} = 4b^2 j_2 j_3 - 4\alpha_2\alpha_3
\end{array}$$

One reads off from (18) the following relation between the two monodromies:

$$F^{H_3^+}_{j_{21},j_{32}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon = e^{i\pi\epsilon b^{-1}\alpha_2} M^{-\frac{1}{2b}}_{\alpha_{21},\alpha_{32}}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_\epsilon \tag{19}$$

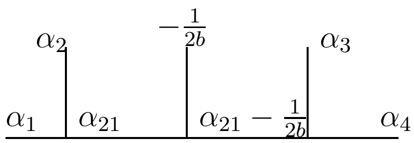
where $\epsilon = \pm 1$ depends wether $Im(x-z)$ is negative or positive. This leads to an ambiguity in the definition of the H_3^+ WZNW fusion matrix, for this reason we shall add to the fusion matrix the subscript ϵ. Nevertheless, this ambiguity does not appear in the bootstrap, where one considers both the holomorphic conformal block and its antiholomorphic counterpart.

2.2 Monodromy of the Liouville 5 point conformal block

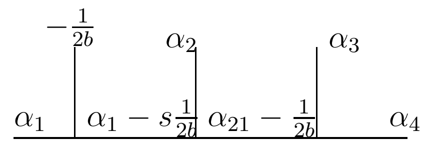
The monodromy of this special 5 point conformal block decomposes in a succession of elementary braiding and fusing transformations.
Let us remember that the braiding is related to the fusion the following way [20]:

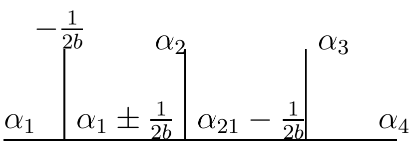
$$B^{-\epsilon}_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc}\alpha_4 & \alpha_2\\ \alpha_3 & \alpha_1\end{array}\right] = e^{i\pi\epsilon(\Delta^L_{\alpha_{21}}+\Delta^L_{\alpha_{32}}-\Delta^L_{\alpha_3}-\Delta^L_{\alpha_4})}F^L_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right] \tag{20}$$

It is then straightforward to obtain the monodromy for this 5 point conformal block thanks to the picture:

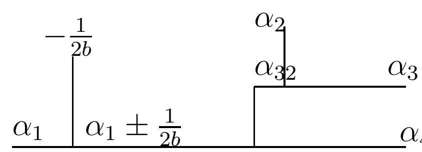


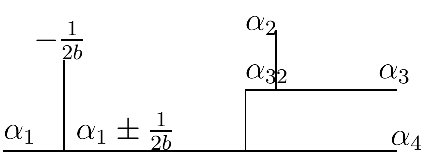
$$= \sum_{s=\pm} B^{-\epsilon}_{\alpha_{21},\alpha_1 - s\frac{1}{2b}}\left[\begin{array}{cc}-\frac{1}{2b} & \alpha_2\\ \alpha_{21}-\frac{1}{2b} & \alpha_1\end{array}\right]$$



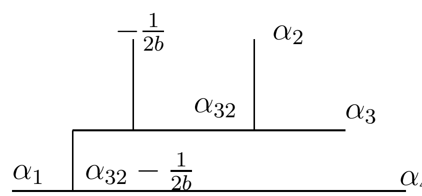


$$= \int d\alpha_{32} F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1 - s\frac{1}{2b}\end{array}\right]$$





$$= F^L_{\alpha_1 - s\frac{1}{2b},\alpha_{32}-\frac{1}{2b}}\left[\begin{array}{cc}\alpha_{32} & -\frac{1}{2b}\\ \alpha_4 & \alpha_1\end{array}\right]$$



where the first picture represents the s-channel 5 point conformal block, and the last one the t-channel conformal block.
Collecting the terms together, we get:

$$\begin{aligned}
&M^{-\frac{1}{2b}}_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_\epsilon = \\
&\quad e^{i\pi\epsilon\frac{1}{b}(-\alpha_{21}+\alpha_1)}F^L_{-+}\left[\begin{array}{cc}\alpha_1 & -\frac{1}{2b}\\ \alpha_2 & \alpha_{21}-\frac{1}{2b}\end{array}\right]F^L_{++}\left[\begin{array}{cc}\alpha_{32} & -\frac{1}{2b}\\ \alpha_4 & \alpha_1\end{array}\right]F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1-\frac{1}{2b}\end{array}\right]+\\
&\quad e^{i\pi\epsilon\frac{1}{b}(-\alpha_{21}-\alpha_1+Q)}F^L_{--}\left[\begin{array}{cc}\alpha_1 & -\frac{1}{2b}\\ \alpha_2 & \alpha_{21}-\frac{1}{2b}\end{array}\right]F^L_{-+}\left[\begin{array}{cc}\alpha_{32} & -\frac{1}{2b}\\ \alpha_4 & \alpha_1\end{array}\right]F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1+\frac{1}{2b}\end{array}\right]
\end{aligned} \tag{21}$$

2.3 Fusion matrix of the H_3^+ WZNW model

The formula for the H_3^+ WZNW fusion matrix follows (see Appendix for this special value of the Liouville fusion coefficient when one of the $\alpha_i, i = 1 \ldots 4$ is equal to $-b^{-1}/2$):

$$F^{H_3^+}_{j_{21},j_{32}} \left[\begin{array}{cc} j_3 & j_2 \\ j_4 & j_1 \end{array} \right]_\epsilon = e^{i\pi\epsilon b^{-1}\alpha_2} M^{-\frac{1}{2b}}_{\alpha_{21},\alpha_{32}} \left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{array} \right]_\epsilon \tag{22}$$

$$\begin{aligned} & F^{H_3^+}_{j_{21},j_{32}} \left[\begin{array}{cc} j_3 & j_2 \\ j_4 & j_1 \end{array} \right]_\epsilon = \\ & \quad \frac{\pi\Gamma(2+b^{-2}+2j_{21})\Gamma(1+2j_{32})}{\sin\pi(j_1+j_2-j_3-j_4)} \times \\ & \quad \Big(\frac{e^{i\pi\epsilon(j_{21}-j_1-j_2)}}{\Gamma(2+b^{-2}+j_{21}+j_1+j_2)\Gamma(j_{21}-j_3-j_4)\Gamma(1+j_{32}-j_1+j_4)\Gamma(1+j_3-j_2+j_{32})} \times \\ & \quad \times F^L_{-bj_{21},-bj_{32}+1/2b} \left[\begin{array}{cc} -bj_3+1/2b & -bj_2 \\ -bj_4+1/2b & -bj_1 \end{array} \right] \\ & \quad - \frac{e^{i\pi\epsilon(j_{21}-j_3-j_4)}}{\Gamma(2+b^{-2}+j_{21}+j_3+j_4)\Gamma(j_{21}-j_1-j_2)\Gamma(1+j_{32}+j_1-j_4)\Gamma(1-j_3+j_2+j_{32})} \times \\ & \quad \times F^L_{-bj_{21},-bj_{32}+1/2b} \left[\begin{array}{cc} -bj_3 & -bj_2+1/2b \\ -bj_4 & -bj_1+1/2b \end{array} \right] \Big) \end{aligned} \tag{23}$$

It is possible to obtain an alternative representation for the H_3^+ fusion matrix: if we consider the pentagone equation (7) in the special case where $\alpha_2 = -\frac{1}{2b}$, and using the fusion rules of [21], then the LFT fusion matrices which contain $\alpha_2 = -\frac{1}{2b}$ are some residues of the general fusion coefficients (4). Their expressions are given in the Appendix A. We then obtain an equation that permits us to reexpress *each* LFT fusion matrix in (21)

$$F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}} \left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 - \frac{1}{2b} \end{array} \right] \quad , \quad F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}} \left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 + \frac{1}{2b} \end{array} \right]$$

in terms of

$$F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}-\frac{1}{2b}} \left[\begin{array}{cc} \alpha_3 & \alpha_2 - \frac{1}{2b} \\ \alpha_4 & \alpha_1 \end{array} \right] \quad \textit{and} \quad F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}-\frac{1}{2b}} \left[\begin{array}{cc} \alpha_3 & \alpha_2 + \frac{1}{2b} \\ \alpha_4 & \alpha_1 \end{array} \right]$$

Rearranging the terms together, one gets the following representation:

$$
\begin{aligned}
&F^{H_3^+}_{j_{21},j_{32}}\left[\begin{array}{cc} j_3 & j_2 \\ j_4 & j_1 \end{array}\right]_\epsilon = \\
&\quad \frac{\pi\Gamma(2+b^{-2}+2j_{21})\Gamma(-2j_{32}-b^{-2}-1)}{\sin\pi(j_1+j_2+j_3+j_4+2+b^{-2})} \times \\
&\quad \Big(\frac{\Gamma^{-1}(2+b^{-2}+j_{21}+j_1+j_2)\Gamma^{-1}(2+b^{-2}+j_{21}+j_3+j_4)}{\Gamma(-1-b^{-2}-j_{32}-j_2-j_3)\Gamma(-1-b^{-2}-j_{32}-j_1-j_4)} \times \\
&\quad \times F^L_{-bj_{21},-bj_{32}}\left[\begin{array}{cc} -bj_3 & -bj_2 \\ -bj_4 & -bj_1 \end{array}\right] \\
&\quad -\frac{e^{-i\pi\epsilon(j_1+j_2+j_3+j_4+2+b^{-2})}}{\Gamma(j_{21}-j_1-j_2)\Gamma(j_{21}-j_3-j_4)\Gamma(1-j_{32}+j_2+j_3)\Gamma(1-j_{32}+j_1+j_4)} \times \\
&\quad \times F^L_{-bj_{21},-bj_{32}}\left[\begin{array}{cc} -bj_3+\frac{1}{2b} & -bj_2+\frac{1}{2b} \\ -bj_4+\frac{1}{2b} & -bj_1+\frac{1}{2b} \end{array}\right]\Big)
\end{aligned} \tag{24}
$$

We can use the same trick again by setting this time $\alpha_3 = -\frac{1}{2b}$ in (7), so that we can reexpress

$$
F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1-\frac{1}{2b} \end{array}\right] \quad , \quad F^L_{\alpha_{21}-\frac{1}{2b},\alpha_{32}}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1+\frac{1}{2b} \end{array}\right]
$$

in terms of

$$
F^L_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4-\frac{1}{2b} & \alpha_1 \end{array}\right] \quad and \quad F^L_{\alpha_{21},\alpha_{32}}\left[\begin{array}{cc} \alpha_3 & \alpha_2 \\ \alpha_4+\frac{1}{2b} & \alpha_1 \end{array}\right]
$$

This gives us a third representation for the H_3^+ fusion matrix:

$$
\begin{aligned}
&F^{H_3^+}_{j_{21},j_{32}}\left[\begin{array}{cc} j_3 & j_2 \\ j_4 & j_1 \end{array}\right]_\epsilon = \\
&\quad \frac{\pi\Gamma(-2j_{21})\Gamma(1+2j_{32})}{\sin\pi(-j_1+j_2-j_3+j_4)} \times \\
&\quad \Big(\frac{e^{i\pi\epsilon(j_{21}+j_{32}-j_1-j_3)}}{\Gamma(-j_{21}-j_1+j_2)\Gamma(-j_{21}-j_3+j_4)\Gamma(1+j_{32}-j_2+j_3)\Gamma(1+j_{32}+j_1-j_4)} \times \\
&\quad \times F^L_{-bj_{21}+\frac{1}{2b},-bj_{32}+\frac{1}{2b}}\left[\begin{array}{cc} -bj_3+\frac{1}{2b} & -bj_2 \\ -bj_4 & -bj_1+\frac{1}{2b} \end{array}\right] \\
&\quad -\frac{e^{i\pi\epsilon(j_{21}+j_{32}-j_2-j_4)}}{\Gamma(-j_{21}+j_1-j_2)\Gamma(j_{21}+j_3-j_4)\Gamma(1+j_{32}+j_2-j_3)\Gamma(1+j_{32}-j_1+j_4)} \times \\
&\quad \times F^L_{-bj_{21}+\frac{1}{2b},-bj_{32}+\frac{1}{2b}}\left[\begin{array}{cc} -bj_3 & -bj_2+\frac{1}{2b} \\ -bj_4+\frac{1}{2b} & -bj_1 \end{array}\right]\Big)
\end{aligned} \tag{25}
$$

Consistency checks and remarks:

1. One might wonder why on the first picture we did not also consider the case where the fusion between α_{21} and $-\frac{1}{2b}$ gives $\alpha_{21}+\frac{1}{2b}$ (this comment also applies for the last picture replacing α_{21} by α_{32}). It is not difficult to see that if we do the same reasoning starting with $\alpha_{21}+\frac{1}{2b}$, we would simply have to replace j_{21} by $-j_{21}-1$ in the expression for the fusion matrix. As we consider j_{21} being of the form $-\frac{1}{2}+is$ with s any real number, it is enough to consider only the case where the result of the fusion is $\alpha_{21}-\frac{1}{2b}$ (resp. $\alpha_{32}-\frac{1}{2b}$).

2. The known cases where $j_2 = 1/2$, $j_2 = 1/2b^2$ and $j_2 = -k/2$ can be found in the appendix B.

3. The $SU(2)$ WZNW model is obtained by substituting b by ib and by giving the spins j positive half integer or integer values. One also has to substitute to Liouville theory the $(k+2,1)$ Minimal Model with central charge

$$c = 1 - \frac{6(p-q)^2}{pq}, \qquad p = k+2, \quad q = 1.$$

The second term of the sum in (24) always vanishes as the fusion rules in the $SU(2)$ WZNW model are such that

$$\frac{1}{\Gamma(j_{21}-j_1-j_2)} = \frac{1}{\Gamma(-n)} = 0$$

with n some positive integer. Only the first term of the sum remains. Then one uses the fusion rules $j_{21} = j_1+j_2-n$ and $j_{32} = j_3+j_2-m$ to rewrite the prefactor in front of the Liouville fusion matrix. This gives the (ϵ independant) result:

$$\begin{aligned} & F^{SU(2)}_{j_{21},j_{32}} \begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix} = \\ & = \frac{\Gamma(2+b^{-2}+2j_{21})\Gamma(2+b^{-2}+j_{32}+j_3+j_2)\Gamma(2+b^{-2}+j_{32}+j_1+j_4)}{\Gamma(2+b^{-2}+2j_{32})\Gamma(2+b^{-2}+j_{21}+j_1+j_2)\Gamma(2+b^{-2}+j_{21}+j_3+j_4)} \times \\ & \times \quad F^{MM}_{-bj_{21},-bj_{32}} \begin{bmatrix} -bj_3 & -bj_2 \\ -bj_4 & -bj_1 \end{bmatrix} \end{aligned} \tag{26}$$

4. It is straightforward to check the following symmetry properties

$$F^{H_3^+}_{j_{21},j_{32}} \begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon = F^{H_3^+}_{j_{21},j_{32}} \begin{bmatrix} j_2 & j_3 \\ j_1 & j_4 \end{bmatrix}_\epsilon = F^{H_3^+}_{j_{21},j_{32}} \begin{bmatrix} j_4 & j_1 \\ j_3 & j_2 \end{bmatrix}_\epsilon \tag{27}$$

using the fact that these properties hold for the Liouville fusion matrix.

5. Using the invariance of the LFT fusion matrix w.r.t. conformal weights, one can notice the following additional symmetry:

$$F^{H_3^+}_{j_{21},j_{32}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon = e^{-i\pi\epsilon(j_1+j_2+j_3+j_4+2+b^{-2})} F^{H_3^+}_{j_{21},j_{32}}\begin{bmatrix} \tilde{j}_3 & \tilde{j}_2 \\ \tilde{j}_4 & \tilde{j}_1 \end{bmatrix}_\epsilon \tag{28}$$

where $\tilde{j} \equiv -j - \frac{k}{2} \equiv -j - 1 - \frac{1}{2b^2}$, as well as:

$$F^{H_3^+}_{j_{21},j_{32}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon = e^{i\pi\epsilon(j_{21}-j_1-j_2)} F^{H_3^+}_{j_{21},\tilde{j}_{32}}\begin{bmatrix} \tilde{j}_3 & j_2 \\ \tilde{j}_4 & j_1 \end{bmatrix}_{-\epsilon} \tag{29}$$

$$F^{H_3^+}_{j_{21},j_{32}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon = e^{i\pi\epsilon(j_{32}-j_2-j_3)} F^{H_3^+}_{\tilde{j}_{21},j_{32}}\begin{bmatrix} j_3 & j_2 \\ \tilde{j}_4 & \tilde{j}_1 \end{bmatrix}_{-\epsilon} \tag{30}$$

These relations also exist in the $SU(2)$ WZWN model (see for example equ.(98) of [22]). The difference is that in the $SU(2)$ case, $e^{i\pi\epsilon(\dots)}$ is replaced by $(-1)^{(\dots)}$: the reason for this is that there is no ϵ dependence in the $SU(2)$ case, as we saw above.

6. If $Re(j_i), i = 1\dots4$ are close enough to $-\frac{1}{2}$, one is still in the range where both the Liouville fusion matrix that appear in the above formula remain holomorphic.

7. Hexagonal equation:
It is proven that the LFT fusion matrix satisfies polynomial equations [9]. We can use this fact to derive the following equation for the monodromy of a 5 point conformal block in LFT:

$$M^{-\frac{1}{2b}}_{\alpha_{21},\beta}\begin{bmatrix} \alpha_4 & \alpha_2 \\ \alpha_3 & \alpha_1 \end{bmatrix}_\epsilon e^{i\pi\epsilon(\Delta^L_{\alpha_1}+\Delta^L_{\alpha_2}+\Delta^L_{\alpha_3}+\Delta^L_{\alpha_4}-\Delta^L_{\alpha_{21}-\frac{1}{2b}}-\Delta^L_\beta)} =$$
$$\int_{\frac{Q}{2}-i\infty}^{\frac{Q}{2}+i\infty} d\alpha_{32} M^{-\frac{1}{2b}}_{\alpha_{21},\alpha_{32}}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_\epsilon M^{-\frac{1}{2b}}_{\alpha_{32}-\frac{1}{2b},\beta}\begin{bmatrix} \alpha_2 & \alpha_4 \\ \alpha_3 & \alpha_1 \end{bmatrix}_\epsilon e^{i\pi\epsilon\Delta^L_{\alpha_{32}-\frac{1}{2b}}} \tag{31}$$

As the monodromy for a 5 point function in LFT depends on braiding (*i.e.* depends on ϵ), it fixes the ϵ appearing in the exponential, which should be the same as the one parametrizing the LFT monodromy.
It is possible to derive the relation

$$M^{-\frac{1}{2b}}_{\alpha_{32}-\frac{1}{2b},\beta}\begin{bmatrix} \alpha_2 & \alpha_4 \\ \alpha_3 & \alpha_1 \end{bmatrix}_\epsilon = M^{-\frac{1}{2b}}_{\alpha_{32},\beta}\begin{bmatrix} \alpha_4 & \alpha_2 \\ \alpha_1 & \alpha_3 \end{bmatrix}_\epsilon e^{i\pi\epsilon b^{-1}\beta} \tag{32}$$

this leads us to the hexagonal equation for $F^{H_3^+}_{j_{21},q}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon$

$$F^{H_3^+}_{j_{21},q}\begin{bmatrix} j_4 & j_2 \\ j_3 & j_1 \end{bmatrix}_\epsilon e^{i\pi\epsilon(\Delta_{j_1}+\Delta_{j_2}+\Delta_{j_3}+\Delta_{j_4}-\Delta_{j_{21}}-\Delta_q-j_1-j_2-j_3-j_4+j_{21}+q)} =$$
$$\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} dj_{32} F^{H_3^+}_{j_{21},j_{32}}\begin{bmatrix} j_3 & j_2 \\ j_4 & j_1 \end{bmatrix}_\epsilon F^{H_3^+}_{j_{32},q}\begin{bmatrix} j_2 & j_4 \\ j_3 & j_1 \end{bmatrix}_\epsilon e^{i\pi\epsilon(\Delta_{j_{32}}-j_{32})} \tag{33}$$

where β and q are related by $\beta = -bq + \frac{1}{2b}$.
Let us note that the situation here seems to be different from the already known cases (rational CFT's and LFT): appearently, the fusion matrix, for ϵ given, satisfies one hexagonal equation *only*, and not two. But as ϵ can take two values +1 and -1, we indeed have two hexagonal equations in this model.

8. Pentagonal equation:
   I don't have much to say about it as I don't know how to prove it.

9. Let us introduce for short $\delta = j_1 + j_2 + j_3 + j_4 + 2 + b^{-2}$, and let us consider for example (28). This equation is a consequence of the following equality between conformal blocks:

$$\mathcal{G}^s_{j_{21}}(j_1, j_2, j_3, j_4|x, z) = (xz^{-1} - 1)^{\delta} x^{2(j_1+j_2+1+\frac{1}{2b^2})}(1-x)^{2(j_3+j_2+1+\frac{1}{2b^2})} \times \quad z^{-(j_1+j_2+1+\frac{1}{2b^2})}(1-z)^{-(j_3+j_2+1+\frac{1}{2b^2})} \quad \mathcal{G}^s_{j_{21}}(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3, \tilde{j}_4|x, z) \tag{34}$$

We now insert this expression into a four point function

$$\Phi_{j_4,j_3,j_2,j_1}(x, \bar{x}|z, \bar{z}) = = \int_{-\frac{1}{2}+i\mathbb{R}} dj_{21} B(j_{21}) C^{H_3^+}(j_4, j_3, j_{21}) C^{H_3^+}(j_{21}, j_2, j_1) |\mathcal{G}^s_{j_{21}}(j_1, j_2, j_3, j_4|x, z)|^2 \tag{35}$$

We can rewrite this equation thanks to the following property of the three point function

$$C^{H_3^+}(j_3, \tilde{j}_2, \tilde{j}_1) = (\nu(b))^{-b^{-2}} B(j_2) B(j_1) C^{H_3^+}(j_3, j_2, j_1) \tag{36}$$

where $B(j)$ is the two point function.
It follows

$$\Phi_{j_4,j_3,j_2,j_1}(x, \bar{x}|z, \bar{z}) = (B(j_4)B(j_3)B(j_2)B(j_1))^{-1} |z|^{-2(j_1+j_2+1+\frac{1}{2b^2})}|1-z|^{-2(j_3+j_2+1+\frac{1}{2b^2})} \times |xz^{-1} - 1|^{2\delta}|x|^{4(j_1+j_2+1+\frac{1}{2b^2})}|1-x|^{4(j_3+j_2+1+\frac{1}{2b^2})}(\nu(b))^{2b^{-2}} \times \int_{-\frac{1}{2}+i\mathbb{R}} dj_{21} B(j_{21}) C^{H_3^+}(\tilde{j}_4, \tilde{j}_3, j_{21}) C^{H_3^+}(j_{21}, \tilde{j}_2, \tilde{j}_1) |\mathcal{G}^s_{j_{21}}(\tilde{j}_1, \tilde{j}_2, \tilde{j}_3, \tilde{j}_4|x, z)|^2 \tag{37}$$

from which we conclude the following relation holding at the level of correlation functions between $\Phi_{j_4,j_3,j_2,j_1}(x, \bar{x}|z, \bar{z})$ and $\Phi_{\tilde{j}_4,\tilde{j}_3,\tilde{j}_2,\tilde{j}_1}(x, \bar{x}|z, \bar{z})$:

$$\Phi_{j_4,j_3,j_2,j_1}(x, \bar{x}|z, \bar{z}) = (B(j_4)B(j_3)B(j_2)B(j_1))^{-1} \Phi_{\tilde{j}_4,\tilde{j}_3,\tilde{j}_2,\tilde{j}_1}(x, \bar{x}|z, \bar{z}) \times |xz^{-1} - 1|^{2\delta}|x|^{4(j_1+j_2+1+\frac{1}{2b^2})}|1-x|^{4(j_3+j_2+1+\frac{1}{2b^2})}(\nu(b))^{2b^{-2}} \times |z|^{-2(j_1+j_2+1+\frac{1}{2b^2})}|1-z|^{-2(j_3+j_2+1+\frac{1}{2b^2})}. \tag{38}$$

3 Study of the singular behavior $x \to z$

Let us denote $\psi^{\alpha_2}_{\alpha_1,\alpha_{21}}(z)$ the Liouville chiral vertex operators. They satisfy the operator product expansion:

$$\psi^{\alpha_2}_{\alpha_1,\alpha_{21}}(z)\psi^{-\frac{1}{2b}}_{\alpha_{21},\alpha_{21}-\frac{1}{2b}}(x) \sim_{x\to z} (x-z)^{b^{-1}\alpha_2} F^L_{\alpha_{21},\alpha_2-\frac{1}{2b}}\begin{bmatrix} \alpha_2 & -\frac{1}{2b} \\ \alpha_1 & \alpha_{21}-\frac{1}{2b}\end{bmatrix} \psi^{\alpha_2-\frac{1}{2b}}_{\alpha_1,\alpha_{21}-\frac{1}{2b}}(z) + \\ + \quad (x-z)^{\frac{1}{b}(Q-\alpha_2)} F^L_{\alpha_{21},\alpha_2+\frac{1}{2b}}\begin{bmatrix} \alpha_2 & -\frac{1}{2b} \\ \alpha_1 & \alpha_{21}-\frac{1}{2b}\end{bmatrix} \psi^{\alpha_2+\frac{1}{2b}}_{\alpha_1,\alpha_{21}-\frac{1}{2b}}(z) \quad (39)$$

If we insert this relation into (18), we find that the asymptotic behavior when $x \to z$ of the 4 point conformal block in the H_3^+ WZNW is related to the 4 point conformal block in LFT:

$$\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z) \sim_{x\to z} z^{-\frac{1}{2}\gamma_{12}-b^{-1}\alpha_1}(1-z)^{-\frac{1}{2}\gamma_{13}-b^{-1}\alpha_3}\times \\ \times \quad \Bigg[F^L_{\alpha_{21},\alpha_2-\frac{1}{2b}}\begin{bmatrix} \alpha_2 & -\frac{1}{2b} \\ \alpha_1 & \alpha_{21}-\frac{1}{2b}\end{bmatrix} \mathcal{F}^s_{\alpha_{21}-\frac{1}{2b}}(\alpha_1,\alpha_2-\frac{1}{2b},\alpha_3,\alpha_4|z) \\ + \quad (x-z)^{\frac{1}{b}(-2\alpha_2+Q)} F^L_{\alpha_{21},\alpha_2+\frac{1}{2b}}\begin{bmatrix} \alpha_2 & -\frac{1}{2b} \\ \alpha_1 & \alpha_{21}-\frac{1}{2b}\end{bmatrix} \mathcal{F}^s_{\alpha_{21}-\frac{1}{2b}}(\alpha_1,\alpha_2+\frac{1}{2b},\alpha_3,\alpha_4|z)\Bigg] \quad (40)$$

The first term of the sum is regular at $x = z$, whereas the second one is singular.
It is easy to see that the Liouville conformal block $\mathcal{F}^s_{\alpha_{21}-\frac{1}{2b}}(\alpha_1,\alpha_2-\frac{1}{2b},\alpha_3,\alpha_4|z)$ has the same monodromy as $\mathcal{F}^s_{-bj_{21}}(-bj_1,-bj_2,-bj_3,-bj_4|z)$; if we then study the behavior when $z \to 0$ of the first Liouville conformal block multiplied by the spatial factor in front of the bracket, we then see that we have indeed the equality

$$\mathcal{F}^s_{-bj_{21}}(-bj_1,-bj_2,-bj_3,-bj_4|z) = \\ z^{-\frac{1}{2}\gamma_{12}-b^{-1}\alpha_1}(1-z)^{-\frac{1}{2}\gamma_{13}-b^{-1}\alpha_3}\mathcal{F}^s_{\alpha_{21}-\frac{1}{2b}}(\alpha_1,\alpha_2-\frac{1}{2b},\alpha_3,\alpha_4|z) \quad (41)$$

A similar study for the second Liouville conformal block of the sum allows us to rewrite (40) as

$$\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z) \\ \sim_{x\to z} \Bigg[\frac{\Gamma(2+b^{-2}+2j_{21})\Gamma(2+b^{-2}+j_1+j_2+j_3+j_4)}{\Gamma(2+b^{-2}+j_{21}+j_1+j_2)\Gamma(2+b^{-2}+j_{21}+j_3+j_4)}\mathcal{F}^s_{-bj_{21}}(-bj_1,-bj_2,-bj_3,-bj_4|z) + \\ + \frac{\Gamma(2+b^{-2}+2j_{21})\Gamma(-2-b^{-2}-j_1-j_2-j_3-j_4)}{\Gamma(j_{21}-j_1-j_2)\Gamma(j_{21}-j_3-j_4)}\mathcal{F}^s_{-bj_{21}}(-b\tilde{j}_1,-b\tilde{j}_2,-b\tilde{j}_3,-b\tilde{j}_4|z)\times \\ \times \quad (x-z)^{\delta} z^{-\delta+j_1+j_2+1+\frac{1}{2b^2}}(1-z)^{j_3+j_2+1+\frac{1}{2b^2}}\Bigg] \quad (42)$$

Remarks:

1. It is straightforward to check the property

$$\mathcal{G}^s_{j_{21}}(\tilde{j}_1,\tilde{j}_2,\tilde{j}_3,\tilde{j}_4|x,z) \sim_{x\to z} \\ (x-z)^{-\delta} z^{\delta-(j_1+j_2+1+\frac{1}{2b^2})}(1-z)^{-(j_2+j_3+1+\frac{1}{2b^2})}\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z) \tag{43}$$

We recover here a straighforward consequence of equation (34).

2. Let us consider again the case of the $SU(2)$ WZNW model: for the same reason we mentionned previously, the factor

$$\frac{1}{\Gamma(j_{21}-j_1-j_2)}$$

makes the singular term of (42) vanish. Then one sees that up to a normalization of the chiral vertex operators preserving the polynomial equations, $\mathcal{G}(z)$ is nothing but the conformal block

$$\mathcal{F}^s_{-bj_{21}}(-bj_1,-bj_2,-bj_3,-bj_4|z)$$

of the $(k+2,1)$ minimal model, as discussed in [23].

3. Degenerate representations and fusion rules for $\hat{sl}(2)$ are well known [24]; in appendix C we show how this relation permits us to recover them.

4. It is instructive to use relation (42) to express the behavior when $x \to z$ of a correlation function in H_3^+ in terms of *two* correlation functions in Liouville field theory.
We consider

$$\Phi_{j_4,j_3,j_2,j_1}(x,\bar{x}|z,\bar{z}) \\ = \int_{\mathcal{D}} dj_{21} B(j_{21}) C^{H_3^+}(j_4,j_3,j_{21}) C^{H_3^+}(j_{21},j_2,j_1) |\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z)|^2, \tag{44}$$

the external spins are such that $Re(j_i), i = 1,\dots 4$ are close enough to $-\frac{1}{2}$, and $\mathcal{D} = -\frac{1}{2}+i\mathbb{R}$. Inserting (42), one then sees that the prefactor in front the conformal block
$|\mathcal{F}^s_{-bj_{21}}(-bj_1,-bj_2,-bj_3,-bj_4|z)|^2$ multiplied by the H_3^+ three point function recombines to give the Liouville three point function

$$C^L(-bj_4,-bj_3,Q+bj_{21})C^L(-bj_{21},-bj_2,-bj_1).$$

Similarly the prefactor of the conformal block $|\mathcal{F}^s_{-bj_{21}}(-b\tilde{j}_1, -b\tilde{j}_2, -b\tilde{j}_3, -b\tilde{j}_4|z)|^2$ (using the property that the Liouville conformal blocks depend on the Liouville conformal weights) multiplied by the H_3^+ three point function recombines to give the Liouville three point function

$$(B(j_1)B(j_2)B(j_3)B(j_4))^{-1}\, C^L(-b\tilde{j}_4, -b\tilde{j}_3, Q + bj_{21})C^L(-bj_{21}, -b\tilde{j}_2, -b\tilde{j}_1)$$

We have

$$\begin{aligned}
&\Phi_{j_4,j_3,j_2,j_1}(x,\bar{x}|z,\bar{z}) \sim_{x\to z} \\
&\quad a\int_{\mathcal{D}} dj_{21} C^L(-bj_4, -bj_3, Q + bj_{21})C^L(-bj_{21}, -bj_2, -bj_1)|\mathcal{F}^s_{-bj_{21}}(z)|^2 + \\
&\quad + \quad |x-z|^{2\delta}|z|^{-2\delta+2(j_1+j_2+1+\frac{1}{2b^2})}|1-z|^{2(j_3+j_2+1+\frac{1}{2b^2})} \times \quad (B(j_1)B(j_2)B(j_3)B(j_4))^{-1} \times \\
&\quad \times \quad b\int_{\mathcal{D}} dj_{21} C^L(-b\tilde{j}_4, -b\tilde{j}_3, Q + bj_{21})C^L(-bj_{21}, -b\tilde{j}_2, -b\tilde{j}_1)|\tilde{\mathcal{F}}^s_{-bj_{21}}(z)|^2
\end{aligned} \tag{45}$$

with $a = \gamma(j_1+j_2+j_3+j_4+2+b^{-2})$, $\quad b = \gamma(-j_1-j_2-j_3-j_4-2-b^{-2})$. A LFT four point correlation function should factorize over the domain $\mathcal{D}' = -\frac{1}{2}-\frac{1}{2b^2}+i\mathbb{R}$; so we have to deform the contour of integration from $\mathcal{D} = -\frac{1}{2} + i\mathbb{R}$ to $\mathcal{D}' = -\frac{1}{2} - \frac{1}{2b^2} + i\mathbb{R}$ in order to rewrite the expression in terms of correlation functions in LFT. While we deform the contour, we pick up a finite number of those poles $-bj_p$ which are in the range $\frac{b}{2} < Re(-bj_p) < \frac{Q}{2}$, that come from the Liouville three point functions of the regular term (there are no poles coming from the Liouville three point functions of the singular term in this case). It would be of course possible to give whatever values we like for the external spins (for example we could consider them to be real), then the residues that would be picked up depend on a case by case study, as the poles j_p depend on the values of the external spins $j_1 \dots j_4$.

Hence we can rewrite the behavior of the 4 point function in the H_3^+ WZNW model in terms of 4 point functions in LFT:

$$\begin{aligned}
\Phi_{j_4,j_3,j_2,j_1}(x,\bar{x}|z,\bar{z}) &\sim_{x\to z} a\mathcal{V}_{-bj_4,-bj_3,-bj_2,-bj_1}(z,\bar{z}) + b\mathcal{V}_{-b\tilde{j}_4,-b\tilde{j}_3,-b\tilde{j}_2,-b\tilde{j}_1}(z,\bar{z})\times \\
&(B(j_1)B(j_2)B(j_3)B(j_4))^{-1} \times |x-z|^{2\delta}|z|^{-2\delta+2(j_1+j_2+1+\frac{1}{2b^2})}|1-z|^{2(j_3+j_2+1+\frac{1}{2b^2})} \\
&+(\text{Residues}).
\end{aligned} \tag{46}$$

5. If we make the same analysis as above studying this time the behavior when $x \to 1$ of the conformal block, we find (this is a consequence of a relation already noticed in [25]):

$$\mathcal{G}^s_{j_{21}}(j_1, j_2, j_3, j_4|x, z) \sim_{x\to 1} z^{-j_2}(1-z)^{-j_2}\mathcal{G}^s_{\tilde{j}_{21}}(\tilde{j}_1, j_2, j_3, \tilde{j}_4|\frac{z}{x}, z) \tag{47}$$

It is of course possible to make an analysis similar to the one made above to get some relation at the level of correlation functions.

Concluding remarks

There are several points that deserve further study:

- It remains of course to understand precisely the physical meaning of the singularity of the conformal blocks when $x \sim z$.
- It is straightforward to construct in the case of the spherical branes [2] the boundary three point function along the lines of [8]: the normalization for the boundary operators was computed in [2], and the fusion matrix in this case is the one of equation (26). As for the AdS_2 branes, it seems to be a problem to construct a cyclic invariant boundary three point function, as well as to recover the boundary two point function of [2].
- It is worth trying to generalize this method to the supersymmetric case: this could maybe lead to a proof at the level of correlation functions for the duality between N=2 Liouville and the superconformal $SL(2)/U(1)$ model [26]; maybe the results presented here work can also help at proving rigorously the duality conjectured by [27] between the sin-Liouville theory and $SL(2)/U(1)$ WZNW model.
- It would be very interesting to build the fusion matrix as a 6j symbol of a quantum group. The proposal of [12] is the "pair" $U_q(sl(2)), U_{q'}(osp(1|2))$. In particular, such a construction would ensure the validity of the pentagonal equation. We hope to be able to say more about this problem in the future.

Acknowledgments

It is a pleasure to thank V.A.Fateev, I.Kostov, G.Mennessier, V.Schomerus, Ch.Schweigert, Al.B.Zamolodchikov and J.-B. Zuber for discussions and interest in this work.
Work supported by EU under contract HPRN-CT-2000-00122.

Appendix A. Some residues of the Liouville fusion matrix

It is well known that in the case where one of $\alpha_1, \ldots, \alpha_4$, say α_i equals $-\frac{n}{2}b - \frac{m}{2}b^{-1}$ where $n, m \in \mathbb{N}$ and where a triple $(\Delta_{\alpha_4}, \Delta_{\alpha_3}, \Delta_{\alpha_{21}})$, $(\Delta_{\alpha_{21}}, \Delta_{\alpha_2}, \Delta_{\alpha_1})$ which contains Δ_{α_i} satisfies the fusion rules of [21], one will find that the fusion coefficients that multiply the conformal blocks are residues of the general fusion coefficient.
In the case where $\alpha_2 = -\frac{1}{2}b$, the fusion rules are:

$$\begin{cases} \alpha_{21} = \alpha_1 - s\frac{b}{2} \\ \alpha_{32} = \alpha_3 - s'\frac{b}{2} \end{cases}$$

where $s, s' = \pm$.
There are four entries for the fusion matrix in this special case

$$F^L_{\alpha_1 - sb/2, \alpha_3 - s'b/2}\left[\begin{array}{cc} \alpha_3 & -b/2 \\ \alpha_4 & \alpha_1 \end{array}\right] \equiv F^L_{s,s'}$$

which expressions are well known to be:

$$\begin{aligned}
F_{++} &= \frac{\Gamma(b(2\alpha_1 - b))\Gamma(b(b - 2\alpha_3) + 1)}{\Gamma(b(\alpha_1 - \alpha_3 - \alpha_4 + b/2) + 1)\Gamma(b(\alpha_1 - \alpha_3 + \alpha_4 - b/2))} \\
F_{+-} &= \frac{\Gamma(b(2\alpha_1 - b))\Gamma(b(2\alpha_3 - b) - 1)}{\Gamma(b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2) - 1)\Gamma(b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))} \\
F_{-+} &= \frac{\Gamma(2 - b(2\alpha_1 - b))\Gamma(b(b - 2\alpha_3) + 1)}{\Gamma(2 - b(\alpha_1 + \alpha_3 + \alpha_4 - 3b/2))\Gamma(1 - b(\alpha_1 + \alpha_3 - \alpha_4 - b/2))} \\
F_{--} &= \frac{\Gamma(2 - b(2\alpha_1 - b))\Gamma(b(2\alpha_3 - b) - 1)}{\Gamma(b(-\alpha_1 + \alpha_3 + \alpha_4 - b/2))\Gamma(b(-\alpha_1 + \alpha_3 - \alpha_4 + b/2) + 1)}
\end{aligned} \tag{48}$$

The dual case where $\alpha_2 = -b^{-1}/2$ is obtained by substituting b by b^{-1}.

Appendix B. Special cases of the H_3^+ fusion matrix

Degenerate representations and fusion rules are well known for $\hat{sl}(2)$ [24]. We will study three easy cases.

1. $j_2 = 1/2$
   The fusion rules are $j_{21} = j_1 + 1/2 \quad (+), \quad j_{21} = j_1 - 1/2 \quad (-)$. So we are in the case where the H_3^+ fusion matrix reduces to equation (26). It is straightforward to

use the results of Appendix A to get

$$
\begin{aligned}
F^{H_3^+}_{++} &= \frac{\Gamma(b^2(-2j_1-1))\Gamma(b^2(1+2j_3)+1)}{\Gamma(b^2(-j_1+j_3+j_4+1/2)+1)\Gamma(b^2(-j_1+j_3-j_4-1/2))} \\
F^{H_3^+}_{+-} &= \frac{\Gamma(b^2(-2j_1-1))\Gamma(b^2(-2j_3-1))}{\Gamma(-b^2(j_1+j_3+j_4+3/2))\Gamma(b^2(-j_1-j_3+j_4-1/2)} \\
F^{H_3^+}_{-+} &= \frac{\Gamma(1+b^2(1+2j_1))\Gamma(1+b^2(1+2j_3))}{\Gamma(1+b^2(j_1+j_3+j_4+3/2)))\Gamma(1+b^2(j_1+j_3-j_4+1/2))} \\
F^{H_3^+}_{--} &= -\frac{\Gamma(1+b^2(1+2j_1))\Gamma(-b^2(2j_3+1))}{\Gamma(b^2(j_1-j_3-j_4-1/2))\Gamma(b^2(j_1-j_3+j_4+1/2)+1)}
\end{aligned}
\tag{49}
$$

2. $j_2 = 1/2b^2$
   The fusion rules are

$$
j_{21} = j_1 + \frac{1}{2b^2} \quad (+), \quad j_{21} = j_1 - \frac{1}{2b^2} \quad (-), \quad j_{21} = -j_1 - k/2 \quad (\mathrm{x}).
$$

We find it more convenient to work directly with the representation (24). The second LFT fusion matrix of the sum is equal to 1 in the cases where

$$
\begin{aligned}
j_{21} = j_1 - \frac{1}{2b^2}, \quad &\text{or} \quad j_{21} = -j_1 - 1 - \frac{1}{2b^2} \\
j_{32} = j_3 - \frac{1}{2b^2}, \quad &\text{or} \quad j_{32} = -j_3 - 1 - \frac{1}{2b^2},
\end{aligned}
$$

and equal to zero otherwise. It is then straightforward to compute

$$
\begin{aligned}
F^{H_3^+}_{++} &= \frac{\Gamma(-2j_1-b^{-2})\Gamma(2j_3+b^{-2}+1)}{\Gamma(-j_1+j_3+j_4+b^{-2}/2+1)\Gamma(-j_1+j_3-j_4-b^{-2}/2)} \\
F^{H_3^+}_{+-} &= \frac{\Gamma(-2j_1-b^{-2})\Gamma(-2j_3-1)}{\Gamma(-j_1-j_3-j_4-1-b^{-2}/2)\Gamma(-j_1-j_3+j_4-b^{-2}/2)} \\
F^{H_3^+}_{+\mathrm{x}} &= \frac{\Gamma(-2j_1-b^{-2})\Gamma(2j_3+1)\Gamma(-2j_3-1-b^{-2})}{\Gamma(-b^{-2})\Gamma(-j_1+j_3-j_4-b^{-2}/2)\Gamma(-j_1-j_3+j_4-b^{-2}/2)} \\
F^{H_3^+}_{-+} &= \frac{\Gamma(2j_1+2)\Gamma(2j_3+b^{-2}+1)}{\Gamma(j_1+j_3+j_4+b^{-2}/2+2)\Gamma(j_1+j_3-j_4+1+b^{-2}/2)} \\
F^{H_3^+}_{\mathrm{x}+} &= \frac{\Gamma(-2j_1)\Gamma(2j_1+2+b^{-2})\Gamma(2j_3+b^{-2}+1)}{\Gamma(1+b^{-2})\Gamma(-j_1+j_3+j_4+b^{-2}/2+1)\Gamma(j_1+j_3-j_4+1+b^{-2}/2)}
\end{aligned}
\tag{50}
$$

For other cases the second LFT fusion matrix contributes; we find:

$$\begin{aligned}
F^{H_3^+}_{--} &= \frac{e^{-i\pi\epsilon b^{-2}}\Gamma(2j_1+2)\Gamma(-2j_3-1)}{\Gamma(j_1-j_3-j_4-b^{-2}/2)\Gamma(j_1-j_3+j_4+1+b^{-2}/2)} \\
F^{H_3^+}_{-\mathrm{x}} &= -\frac{e^{-i\pi\epsilon(2j_3+b^{-2})}\Gamma(2j_1+2)\Gamma(2j_3+1)\Gamma(-2j_3-1-b^{-2})}{\Gamma(-b^{-2})\Gamma(j_1+j_3+j_4+2+b^{-2}/2)\Gamma(j_1-j_3-j_4-b^{-2}/2)} \\
F^{H_3^+}_{\mathrm{x}-} &= -\frac{e^{-i\pi\epsilon(2j_1+b^{-2})}\Gamma(-2j_1)\Gamma(2j_1+2+b^{-2})\Gamma(-2j_3-1)}{\Gamma(1+b^{-2})\Gamma(-j_1-j_3-j_4-1-b^{-2}/2)\Gamma(j_1-j_3+j_4+1+b^{-2}/2)} \\
F^{H_3^+}_{\mathrm{xx}} &= \Gamma(-2j_1)\Gamma(2j_3+1)\Gamma(2j_1+2+b^{-2})\Gamma(-2j_3-1-b^{-2}) \\
&\times \frac{\sin\pi(-b^{-2})\sin\pi(j_1-j_3-j_4-b^{-2}/2)\sin\pi(-j_1+j_3-j_4-b^{-2}/2)}{\pi^2\sin\pi(j_1+j_3+j_4+2+3b^{-2}/2)} \\
&- \frac{e^{-i\pi\epsilon(j_1+j_3+j_4+2+3b^{-2}/2)}\Gamma(-2j_1)\Gamma(2j_3+1)}{\Gamma(-2j_1-1-b^{-2}/2)\Gamma(2j_3+2+b^{-2}/2)} \qquad (51)
\end{aligned}$$

These coefficients are in agreement with [11] for the choice $\epsilon = 1$, excepted for $F^{H_3^+}_{\mathrm{xx}}$, where there seems to be a slight discrepancy.

3. $j_2 = -k/2$
The fusion rule is $j_{21} = -j_1 - k/2$. The first term of the sum vanishes and the second LFT fusion matrix is equal to one. It remains to evaluate the product of gamma functions; we find:

$$F^{H_3^+} = e^{-i\pi\epsilon(j_1+j_3+j_4+1+\frac{1}{b^2})} \qquad (52)$$

Appendix C. Fusion rules for $\hat{sl}(2)$ algebra at generic level

We present here a way to find degenerate representations and fusion rules for $\hat{sl}(2)$ from the knowledge of the degenerate representations and fusion rules of the Virasoro algebra. We derived in the main text the following asymptotic behavior when $x \to z$ for the conformal blocks of the $SL(2,\mathbb{C})/SU(2)$ WZNW, relating them to two Liouville conformal blocks:

$$\begin{aligned}
&\mathcal{G}^s_{j_{21}}(j_1,j_2,j_3,j_4|x,z) \\
&\quad\sim_{x\to z} \Bigg[\frac{\Gamma(2+b^{-2}+2j_{21})\Gamma(2+b^{-2}+j_1+j_2+j_3+j_4)}{\Gamma(2+b^{-2}+j_{21}+j_1+j_2)\Gamma(2+b^{-2}+j_{21}+j_3+j_4)}\mathcal{F}^s_{-bj_{21}}(-bj_1,-bj_2,-bj_3,-bj_4|z)+ \\
&\quad+\frac{\Gamma(2+b^{-2}+2j_{21})\Gamma(-2-b^{-2}-j_1-j_2-j_3-j_4)}{\Gamma(j_{21}-j_1-j_2)\Gamma(j_{21}-j_3-j_4)}\mathcal{F}^s_{-bj_{21}}(-bJ_1,-bJ_2,-bJ_3,-bJ_4|z)\times \\
&\quad\times\quad (x-z)^{j_1+j_2+j_3+j_4+2+b^{-2}}z^{-j_3-j_4-1-\frac{1}{2b^2}}(1-z)^{j_3+j_2+1+\frac{1}{2b^2}}\Bigg]
\end{aligned} \qquad (53)$$

Although there is no closed form known for the Liouville conformal blocks, they are completely determined by the conformal symmetry. They depend on conformal weights only, *i.e.* are invariant when $-bj_i$ (resp. $-bJ_i$) is changed into $Q+bj_i$ (resp. $Q+bJ_i$).

1. Degenerate representations of the Virasoro algebra.
   It is well known that the case where j_i (resp. J_i) equals $\frac{n}{2}+\frac{m}{2}b^{-2}$ with n,m positive integers, corresponds to a degenerate Virasoro representation $\mathcal{V}_{-bj_i}$ (resp. $\mathcal{V}_{-bJ_i}$). The conformal block $\mathcal{F}^s_{-bj_{21}}$ then only exists for a finite number of values of $-bj_{21}$ [21]:
   $$-bj_{21} = -bj_1 + bj_2 - ub - vb^{-1}. \tag{54}$$
   where u,v are postive integers such that $0 \le u \le n, \quad 0 \le v \le m$.
   It would be equivalent to write $Q+bj_{21}$ instead of $-bj_{21}$ in equation (54) since the Virasoro representations $\mathcal{V}_{-bj}$ and $\mathcal{V}_{Q+bj}$ are equivalent. We will see that this fact will play an important role in the determination of the degenerate representations and fusion rules for $\hat{sl}(2)$.

2. Degenerate representations and fusion rules for $\hat{sl}(2)$.
   Let us call for short F^1 the first Liouville conformal block of the sum (53) and the second one F^2.

   Claim 1 *The spin $j_{m,n}$ that labels the degenerate representation of $\hat{sl}(2)$ $\mathcal{P}_{j_{n,m}}$ are also labels for the degenerate Virasoro representation $\mathcal{V}_{-bj_{n,m}}$ or $\mathcal{V}_{-bJ_{n,m}}$.*

   In other words, it means that $\mathcal{P}_{j_{n,m}}$ is a degenerate representation of $\hat{sl}(2)$ iff:
   $$\begin{array}{lrcl} \text{(a)} & j_{n,m} & = & \frac{n}{2}+\frac{m}{2}b^{-2} \quad \text{or} \\ \text{(b)} & j_{n,m} & = & -\left(\frac{n}{2}+1\right)-\left(\frac{m}{2}+\frac{1}{2}\right)b^{-2}. \end{array} \tag{55}$$
   with n,m positive integers.
   We now provide two rules that will allow us to determine the $\hat{sl}(2)$ fusion rules:

   Rule 1 *If j_2 is such that both $\mathcal{V}_{-bj_2}$ and $\mathcal{V}_{-bJ_2}$ correspond to degenerate Virasoro representations, then the admissible $\hat{sl}(2)$ fusion rules consist of the set of common fusions rules plus $j_{21}=j_1+j_2$ if j_2 is of the form (a), and $j_{21}=-j_1-j_2-2-b^{-2}$ if j_2 is of the form (b).*

   Rule 2 *If j_2 is such that $\mathcal{V}_{-bj_2}$ is a degenerate Virasoro representation and not $\mathcal{V}_{-bJ_2}$ (or converse), then the $\hat{sl}(2)$ fusion rules are such that the factor $\Gamma^{-1}(j_{21}-j_1-j_2)$ in front of F^2 should be equal to zero (resp. $\Gamma^{-1}(2+b^{-2}+j_{21}+j_1+j_2)$ in front of F^1). Note that this case happens only if m=0.*

   We shall start by three easy examples as the generalization is straightforward.

(a) Examples:

i. $j_2 = \frac{1}{2}$
In this case we have $-bj_2 = -\frac{b}{2}$ so the Virasoro representation $\mathcal{V}_{-\frac{b}{2}}$ is degenerate. The fusion rules are:

$$\begin{aligned} -bj_{21} &= -bj_1 - \frac{b}{2}, \quad -bj_{21} = -bj_1 + \frac{b}{2}, \quad \text{or} \\ Q + bj_{21} &= -bj_1 - \frac{b}{2}, \quad Q + bj_{21} = -bj_1 + \frac{b}{2}. \end{aligned} \tag{56}$$

(57)

Let us turn to F^2: $-bJ_2 = -\frac{b}{2} + \frac{1}{2b}$ does not correspond to any degenerate Virasoro representation. We use rule 2 to select the admissible set of fusion rules, and find

$$j_{21} = j_1 + j_2, \quad j_{21} = j_1 - j_2. \tag{58}$$

ii. $j_2 = \frac{1}{2b^2}$
In this case we have $-bj_2 = -\frac{1}{2b}$ so the Virasoro representation $\mathcal{V}_{-\frac{1}{2b}}$ is degenerate. The allowed values for j_{21} are:

$$\begin{aligned} -bj_{21} &= -bj_1 - \frac{1}{2b}, \quad -bj_{21} = -bj_1 + \frac{1}{2b}, \quad \text{or} \\ Q + bj_{21} &= -bj_1 - \frac{1}{2b}, \quad Q + bj_{21} = -bj_1 + \frac{1}{2b}. \end{aligned} \tag{59}$$

As for the second term, we have $-bJ_2 = 0$, which corresponds to the identity representation. The fusion rules are $-bj_{21} = -bj_1 + \frac{1}{2b}$ or $Q + bj_{21} = -bj_1 + \frac{1}{2b}$. The common set of fusion rules consists of

$$-bj_{21} = -bj_1 + \frac{1}{2b}, \quad -bj_{21} = bj_1 - \frac{1}{2b} + Q. \tag{60}$$

According to the rule 1, we should also include $-bj_{21} = -bj_1 - \frac{1}{2b}$.
As a conclusion, we are left with the following three possibilities:

$$-bj_{21} = -bj_1 - \frac{1}{2b}, \quad -bj_{21} = -bj_1 + \frac{1}{2b}, \quad -bj_{21} = bj_1 - \frac{1}{2b} + Q. \tag{61}$$

iii. $j_2 = -\frac{k}{2}$
In this case $-bj_2 = b + \frac{1}{2b}$ does not correspond to any degenerate Virasoro representation, and $-bJ_2 = Q$. As $\mathcal{V}_Q$ and $\mathcal{V}_0$ are equivalent Virasoro representations, we see that $-bJ_2 = Q$ actually corresponds to the identity representation. Hence the fusion rules are $-bj_{21} = -bj_1 + \frac{1}{2b}$ or $-bj_{21} = bj_1 + b + \frac{1}{2b}$. The latter rule is the only acceptable one, as it makes the

term $\Gamma^{-1}(2+b^{-2}+j_{21}+j_1+j_2)$ in front of F^1 vanish.

Let us note that the $\hat{sl}(2)$ representation $\mathcal{P}_{-\frac{k}{2}}$ plays a role very similar to the identity, as the decomposition of its tensor product with an arbitrary representation $\mathcal{P}_j$ gives the representation $\mathcal{P}_{-j-\frac{k}{2}}$ only.

(b) General case:

i. $j_2 = \frac{n}{2} + \frac{m}{2}b^{-2}$, with $n \in \mathbb{N}, m \in \mathbb{N}$.
The allowed values for j_{21} are either

$$\begin{aligned} j_{21} &= j_1 - j_2 + u + vb^{-2} \quad \text{or} \\ j_{21} &= j_2 - j_1 - (u'+1) - (v'+1)b^{-2}, \end{aligned} \tag{62}$$

where $0 \leq u \leq n, \quad 0 \leq v \leq m, \quad 0 \leq u' \leq n, \quad 0 \leq v' \leq m-1.$

ii. $j_2 = -\left(\frac{n}{2}+1\right) - \left(\frac{m}{2}+\frac{1}{2}\right)b^{-2}$, with $n \in \mathbb{N}, m \in \mathbb{N}$.
The allowed values for j_{21} are either

$$\begin{aligned} j_{21} &= j_1 - j_2 - (U+1) - (V+1)b^{-2} \quad \text{or} \\ j_{21} &= j_2 - j_1 + U' + V'b^{-2}, \end{aligned} \tag{63}$$

where $0 \leq U \leq n, \quad 0 \leq V \leq m-1, \quad 0 \leq U' \leq n, \quad 0 \leq V' \leq m.$

These results are in agreement with [24].

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